The Nonlinear Eigenvalue Problem: Part I

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Outline

Two lectures:

- Part I: Mathematical properties of nonlinear eigenproblems (NEPs)
  - Definition and historical aspects
  - Examples and applications
  - Solution structure

- Part II: Numerical methods for NEPs
  - Solvers based on Newton's method
  - Solvers using contour integrals
  - Linear interpolation methods

S. GÜTTEL AND F. TISSEUR, *The nonlinear eigenvalue problem*. Acta Numerica 26:1–94, 2017.

Nonlinear Eigenvalue Problems (NEPs)

Let $F : \Omega \to \mathbb{C}^{n\times n}$ with $\Omega \subseteq \mathbb{C}$ a nonempty open set.

Nonlinear eigenvalue problem: Find scalars λ and nonzero $x, y \in \mathbb{C}^n$ satisfying $F(\lambda)x = 0$ and $y^*F(\lambda) = 0$.

- λ is an e'val, x, y are corresponding right/left e'vecs.
- E'vals are solutions of $p(\lambda) = \det(F(\lambda)) = 0$.
- Algebraic multiplicity of λ is multiplicity of λ as root of p.

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Familiar examples:

- **Standard** eigenvalue problem: $F(\lambda) = \lambda I - A$.
- **Generalized** eigenvalue problem: $F(\lambda) = \lambda B - A$.
- **Quadratic** eigenvalue problem: $F(\lambda) = \lambda^2 M + \lambda D + K$.

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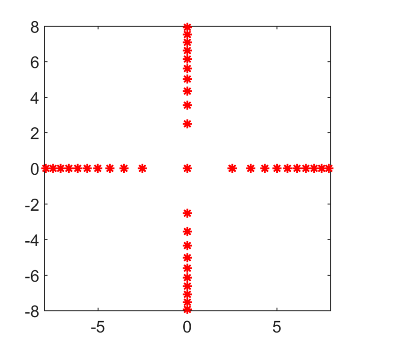
In practice, elements of F can be **polynomial, rational or exponential** functions of λ.

Example 1

$F(z) = \begin{bmatrix} e^{iz^2} & 1 \\ 1 & 1 \end{bmatrix}$ on $\Omega = \mathbb{C}$. Roots of $\det F(z) = e^{iz^2} - 1$ are

$$\lambda_k = \pm\sqrt{2\pi k}, \quad k = 0, \pm 1, \pm 2, \ldots,$$

with $\lambda_0 = 0$ of alg. multiplicity 2 and all others simple.



$$\Omega = \{z \in \mathbb{C} : \\ -8 \le \mathrm{Re}(z) \le 8, \\ -8 \le \mathrm{Im}(z) \le 8\}.$$

Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a left and right e'vec for all of the e'vals!

Remarks

Generally, an NEP can have

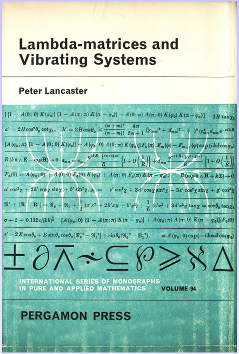
- no eigenvalues at all, e.g., if $F(z) = \exp(z)$,
- finitely many eigenvalues, e.g., if $F(z) = z^3 - 1$,
- countably many eigenvalues, e.g., if $F(z) = \cos(z)$,
- a continuum of eigenvalues, e.g., $F(z) = 0$.

To exclude last case, F is called **regular** if $\det F(z) \not\equiv 0$ on Ω.

Historical Aspects

- In the 1930s, **Frazer, Duncan & Collar** were developing matrix methods for analyzing **flutter** in aircraft.
- Worked in Aerodynamics Division of NPL.
- Developed matrix structural analysis.
- Wrote **Elementary Matrices & Some Applications to Dynamics and Differential Equations, 1938**.
- **Olga Taussky**, in Frazer's group at NPL, 1940s.
  6×6 quadratic eigenvalue problems from flutter in supersonic aircraft.

Historical Aspects (cont.)



- **Peter Lancaster**, English Electric Co., 1950s solved quadratic eigenvalue problems of dimension 2 to 20.

Books

- Lancaster, **Lambda-Matrices and Vibrating Systems**, 1966 (Pergamon), 2002 (Dover).
- Gohberg, Lancaster, Rodman, **Matrix Polynomials**, 1982 (Academic Press), 2009 (SIAM).
- Gohberg, Lancaster, Rodman, **Indefinite Linear Algebra and Applications**, 2005 (Birkhäuser).

NLEVP Toolbox

Collection of Nonlinear Eigenvalue Problems : T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, F. T., TOMS 2013.

- Quadratic, polynomial, rational and other nonlinear eigenproblems.
- Provided in the form of a MATLAB Toolbox.
- Problems from real-life applications + specifically constructed problems.

http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html

Sample of Quadratic Problems

$n \times m$ quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$.

Speaker box (`pep,qep,real,symmetric`).
$n = m = 107$. Finite element model of a speaker box.
$\|M\|_2 = 1$, $\|D\|_2 = 5.7 \times 10^{-2}$, $\|K\|_2 = 1.0 \times 10^7$.

Railtrack (`pep,qep,t-palindromic,sparse`).
$n = m = 1005$. Model of vibration of rail tracks under the excitation of high speed trains. $M = K^T$, $D = D^T$.

Surveillance (`pep,qep,real,nonsquare, nonregular`). $n = 21$, $m = 16$. From calibration of surveillance camera using human body as calibration target.

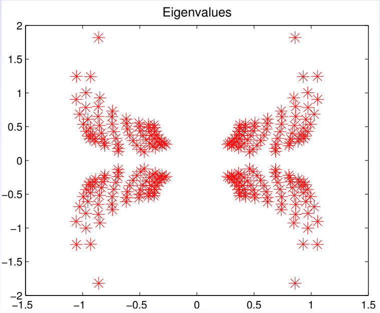
Sample of Higher Degree Problems

$n \times n$ matrix polynomials: $P(\lambda) = \sum_{j=0}^{d} \lambda^j C_j$.

Orr-Sommerfeld(`pep,parameter-dependent`). Quartic arising in spatial stability analysis of Orr-Sommerfeld eq.

Plasma drift(`pep`). Cubic polynomial from modeling of drift instabilities in the plasma edge inside a Tokamak reactor.

Butterfly (`pep,real, T-even, scalable`) quartic matrix polynomial with T-even structure: $P^T(-\lambda) = P(\lambda)$.



Boundary Value Problem (Solov'ëv 2006)

Differential equations with nonlinear boundary conditions

$$-u''(x) = \lambda u(x) \text{ on } [0,1], \quad u(0) = 0, \quad -u'(1) = \phi(\lambda)u(1).$$

Finite element discretization yields NEP

$$F(\lambda)u = (C_1 - \lambda C_2 + \phi(\lambda)C_3)u = 0$$

with $n \times n$ matrices

$$C_1 = n \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad C_2 = \frac{1}{6n} \begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix},$$

$C_3 = e_n e_n^T$, and $e_n = [0, \ldots, 0, 1]^T$.

Sample of Rational/Nonlinear Problems

Loaded string (`rep,real,symmetric,scalable`) rational eigenvalue problem describing eigenvibration of a loaded string.

$$F(\lambda)v = \left(C_1 - \lambda C_2 + \frac{\lambda}{\lambda - \sigma} C_3 \right) v = 0.$$

Gun (`nep,sparse`) nonlinear eigenvalue problem modeling a radio-frequency gun cavity.

$$F(\lambda)v = \left[K - \lambda M + i(\lambda - \sigma_1^2)^{\frac{1}{2}} W_1 + i(\lambda - \sigma_2^2)^{\frac{1}{2}} W_2 \right] v = 0.$$

- C_3, W_1 and W_2 have **low rank**.

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Time delay(`nep,real`) from stability analysis of delay differential equations (DDEs).

$$F(\lambda)v = (\lambda I + A + Be^{-\lambda})v = 0,$$

$\mathrm{Re}(\lambda)$ determines growth of solution.

Other Classes of NEPs

- **Multi-parameter NEPs**

$$F(\lambda, \gamma)v = 0.$$

Arise in stability analysis of parametrized nonlinear wave equations and of delay differential equations. Numerical solution via continuation and solution of an NEP at every iteration.

- **Eigenvector nonlinearities**: find nontrivial solutions of

$$F(V)V = VD,$$

where $n \times n$ F depends on $n \times k$ matrix of e'vecs V, and $k \times k$ diagonal D contains e'vals. [Not considered in this talk.]

More Definitions and Comments

Assumption: Holomorphic NEP

We assume that F is holomorphic on a domain $\Omega \subseteq \mathbb{C}$, denoted by $F \in H(\Omega, \mathbb{C}^{n \times n})$.

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We assume that F is holomorphic on a domain $\Omega \subseteq \mathbb{C}$, denoted by $F \in H(\Omega, \mathbb{C}^{n\times n})$.

- **Algebraic multiplicity** of isolated e'val λ of F is finite but not necessarily bounded by n.
- **Geometric multiplicity** of $\lambda := \dim(\mathrm{null} F(\lambda))$.
- $v_0, v_1, \ldots, v_{m-1}$ are **generalized e'vecs** (Jordan chain) of F with e'val λ if $\sum_{k=0}^{j} \frac{F^{(k)}(\lambda)}{k!} v_{j-k} = 0$, $0 \le j \le m-1$.
  - v_0 is an e'vec of F.
  - The v_i are not necessarily linearly independent.
  - $v_j = 0$, $j \neq 0$ can happen!
  - Maximal length m of Jordan chain starting at v_0 is **partial multiplicity**.

Factorization: Smith Form

Theorem (Smith form)

*Let $F \in H(\Omega, \mathbb{C}^{n\times n})$ be regular on a nonempty domain Ω. Let λ_i ($i = 1, 2, \dots$) be the distinct e'vals of F in Ω with partial mult $m_{i,1} \ge m_{i,2} \ge \cdots \ge m_{i,d_i}$. Then there exists a **global Smith form** $F(z) = P(z)D(z)Q(z)$, $z \in \Omega$, with unimodular $P, Q \in H(\Omega, \mathbb{C}^{n\times n})$, and $D(z) = \mathrm{diag}(\delta_1(z), \delta_2(z), \dots, \delta_n(z))$, where*

$$\delta_j(z) = h_j(z) \prod_{i=1,2,\dots} (z - \lambda_i)^{m_{i,j}}, \quad j = 1, \dots, n,$$

$m_{i,j} = 0$ if $j > d_i$ and each $h_j \in H(\Omega, \mathbb{C})$ is free of roots on Ω.

Resolvent is given by

$$F(z)^{-1} = Q(z)D(z)^{-1}P(z) = \sum_{j=1}^{n} \delta_j(z)^{-1} q_j(z) p_j(z)^*.$$

Keldysh's Theorem

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Theorem (Keldysh 1951)

Let $\lambda \in \Lambda(F)$ have alg multiplicity m and geom multiplicity d. Then there are $n \times m$ matrices V and W and an $m \times m$ Jordan matrix J with eigenvalue λ of partial multiplicities $m_1 \geq m_2 \geq \cdots \geq m_d$, $\sum_{i=1}^{d} m_i = m$, such that

$$F(z)^{-1} = V(zI - J)^{-1}W^* + R(z)$$

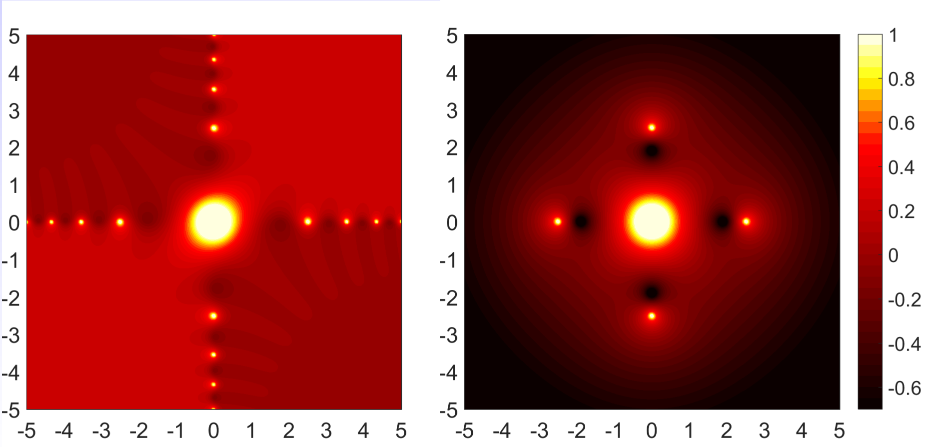
in some open neighborhood $\mathcal{U}$ of λ, with $R \in H(\mathcal{U}, \mathbb{C}^{n\times n})$.

R. MENNICKEN AND M. MÖLLER. *Non-self-adjoint boundary eigenvalue problems,* Elsevier Science, 2003.

Spectral Portraits $z \mapsto \log_{10} \|F(z)^{-1}\|_2$

$F(z) = \begin{bmatrix} e^{iz^2} & 1 \\ 1 & 1 \end{bmatrix}$. Near $\lambda = 0, \pm\sqrt{2\pi}, \pm\mathrm{i}\sqrt{2\pi}$, Keldysh's thm says that

$F(z)^{-1} \approx V(zI - J)^{-1}W^*$, $J = \mathrm{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \pm\sqrt{2\pi}, \pm\mathrm{i}\sqrt{2\pi}\right)$.



Invariant Pairs

Can always write $F \in H(\Omega, \mathbb{C}^{n\times n})$ in "split form"

$$F(z) = f_1(z)C_1 + f_2(z)C_2 + \cdots + f_\ell(z)C_\ell, \quad \ell \le n^2.$$

Definition (invariant pair)

A pair $(V, \Lambda) \in \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times m}$ is called an **invariant pair for** $F \in H(\Omega, \mathbb{C}^{n\times n})$ if

$$C_1 V f_1(\Lambda) + C_2 V f_2(\Lambda) + \cdots + C_\ell V f_\ell(\Lambda) = 0. \qquad (*)$$

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- Beyn et al. (2011) introduce (V, Λ) via Cauchy integral.
- Generalizes to NEPs invariant subspaces for $A \in \mathbb{C}^{n\times n}$:

$F(z) = zI - A$ so that $f_1(z) = z$, $f_2(z) = 1$, $C_1 = I$, $C_2 = -A$.
$(*) \iff AV = V\Lambda$, i.e., V is an invariant subspace for A.

- Generalizes standard/Jordan pairs for matrix poly.

Complete Invariant Pair

Invariant pairs may contain redundant information: if (V, Λ) is an invariant pair so is $([V, V], \mathrm{diag}(\Lambda, \Lambda))$.

$(V, \Lambda) \in \mathbb{C}^{n \times m} \times \mathbb{C}^{m \times m}$ is **minimal** if there is $p > 0$ s.t.

$$\mathrm{rank} \begin{bmatrix} V \\ V\Lambda \\ \vdots \\ V\Lambda^{p-1} \end{bmatrix} = m.$$

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See Kressner's block Newton method (2009) and Efferberger's deflation strategy (2013).

Hermitian NEPs

$F(z)$ is said to be **Hermitian** if $F(\bar{z}) = F(z)^*$ for all $z \in \mathbb{C}$.

- E'vals are either real or come in pairs $(\lambda, \bar{\lambda})$.
- If v and w are right and left e'vecs of F with e'val $\lambda \in \mathbb{C}$, then w and v are right and left e'vecs for the e'val $\bar{\lambda}$.
- If $\lambda \in \mathbb{R}$ with right e'vec v, then v is a left e'vec for λ.

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Do **variational principles** for real eigenvalues of Hermitian NEPs exist?

Yes but under some assumptions.

Assumptions (A1)–(A3)

(A1) $F : \mathbb{R} \supseteq \mathbb{I} \to \mathbb{C}^{n\times n}$ is Hermitian and continuously differentiable on open real interval $\mathbb{I}$.

(A2) For every $x \in \mathbb{C}^n \setminus \{0\}$, the real nonlinear equation

$$x^* F\big(p(x)\big) x = 0 \tag{1}$$

has at most one real solution $p(x) \in \mathbb{I}$.

(1) defines implicitly the **generalized Rayleigh functional** p on some open subset $\mathbb{K}(p) \subseteq \mathbb{C}^n \setminus \{0\}$. When $\mathbb{K}(p) \equiv \mathbb{C}^n \setminus \{0\}$, the NEP $F(\lambda)v = 0$ is called **overdamped**.

(A3) For every $x \in \mathbb{K}(p)$ and any $z \in \mathbb{I}$ such that $z \neq p(x)$,

$$\big(z - p(x)\big)\big(x^* F(z) x\big) > 0.$$

When F is overdamped, (A3) holds if $x^* F'(p(x))x > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$.

Nonlinear Variational Principle

An e'val $\lambda \in \mathbb{I}$ of F is a ***k*th eigenvalue** if $\mu = 0$ is the kth largest eigenvalue of the Hermitian matrix $F(\lambda)$.

Theorem (Hadeler 1968)

Assume that F satisfies (A1)–(A3). Then F has at most n e'vals in $\mathbb{I}$. Moreover, if λ_k is a kth eigenvalue of F, then

$$\lambda_k = \min_{\substack{V \in \mathbb{S}_k \\ V \cap \mathbb{K}(p) \neq \emptyset}} \; \max_{\substack{x \in V \cap \mathbb{K}(p) \\ x \neq 0}} p(x) \in \mathbb{I},$$

$$\lambda_k = \max_{\substack{V \in \mathbb{S}_{k-1} \\ V^\perp \cap \mathbb{K}(p) \neq \emptyset}} \; \min_{\substack{x \in V^\perp \cap \mathbb{K}(p) \\ x \neq 0}} p(x) \in \mathbb{I},$$

where $\mathbb{S}_j$ is the set of all subspaces of $\mathbb{C}^n$ of dimension j.

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Used to develop safegarded iteration (Werner 1970).
Preconditioned CG method (Szyld and Xue 2015).

Eigenvalue Localization

Bindel & Hood (2013) provide a Gershgorin's thm for NEPs.

Theorem

Let $F(z) = D(z) + E(z)$ with $D, E \in H(\Omega, \mathbb{C}^{n\times n})$ and D diagonal. Then for any $0 \le \alpha \le 1$,

$$\Lambda(F) \subset \bigcup_{j=1}^{n} \mathbb{G}_j^{\alpha},$$

*where $\mathbb{G}_j^{\alpha}$ is the **jth generalized Gershgorin region***

$$\mathbb{G}_j^{\alpha} = \left\{ z \in \Omega : \ |d_{jj}(z)| \le r_j(z)^{\alpha} c_j(z)^{1-\alpha} \right\}$$

and $r_j(z) = \sum_{k=1}^{n} |e_{jk}(z)|$, $c_j(z) = \sum_{k=1}^{n} |e_{kj}(z)|$.

Sensitivity of Eigenvalues

Assume $\lambda \neq 0$ is a simple e'val with right and left e'vecs v and w. Define **normwise relative condition number of λ**:

$$\kappa(\lambda, F) = \limsup_{\varepsilon \to 0} \left\{ \frac{|\delta\lambda|}{\varepsilon|\lambda|} : \big(F(\lambda+\delta\lambda) + \Delta F(\lambda+\delta\lambda)\big)(v+\delta v) = 0, \right.$$
$$\left. \|\Delta C_j\|_2 \leq \varepsilon \|C_j\|_2,\ j = 1, \dots, \ell \right\},$$

where $\Delta F(z) = f_1(z)\Delta C_1 + f_2(z)\Delta C_2 + \cdots + f_\ell(z)\Delta C_\ell$.
Can show that

$$\kappa(\lambda, F) = \frac{\left(\sum_{j=1}^{\ell} \|C_j\|_2 |f_j(\lambda)|\right) \|v\|_2 \|w\|_2}{|\lambda| |w^* F'(\lambda) v|}.$$

Example 1 (Cont.)

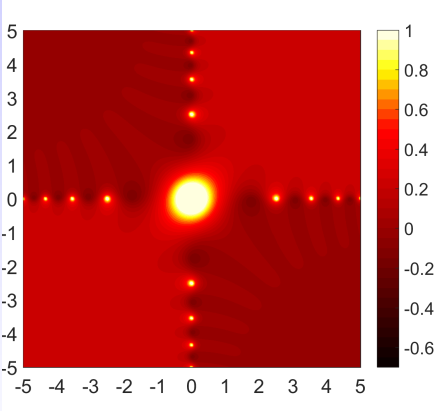
$$F(z) = e^{iz^2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} =: f_1(z)C_1 + f_2(z)C_2.$$

Can show nonzero e'vals of F, $\lambda_k = \pm\sqrt{2\pi k}$, $k = \pm 1, \pm 2, \ldots$, have small condition numbers

$$\kappa(\lambda_k, F) = \frac{1 + \sqrt{1 + (3 + 5^{1/2})/2}}{2\pi|k|}$$

that get smaller as k increases in modulus.

Spectral portrait, $z \mapsto \log_{10} \|F(z)^{-1}\|_2$



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